

## Boundary $K$ -supermatrices for the one-dimensional small-polaron open chain

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1996 J. Phys. A: Math. Gen. 29 L607

(<http://iopscience.iop.org/0305-4470/29/23/005>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.68

The article was downloaded on 02/06/2010 at 03:16

Please note that [terms and conditions apply](#).

LETTER TO THE EDITOR

## Boundary $K$ -supermatrices for the one-dimensional small-polaron open chain

Huan-Qiang Zhou†

CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China, and  
Department of Physics, Chongqing University Chongqing, Sichuan 630044, People's Republic  
of China‡

Received 21 February 1996

**Abstract.** The Lax pair for the one-dimensional small-polaron open chain is explicitly constructed. From this the general boundary  $K$ -supermatrices are found. Our construction provides a direct demonstration for the integrability of the system.

In the last decade, much attention has been paid to the study of completely integrable lattice spin open chains [1–8]. As was shown by Sklyanin [1], there is a variant of the usual formalism of the quantum inverse scattering method (QISM) [9–11], which may be used to describe systems on a finite interval with independent boundary conditions on each end. Central to his approach is the introduction of an algebraic structure called the reflection equations (RE) [12]. Although Sklyanin's argument was carried out only for the  $P$  and  $T$  invariant  $R$ -matrices, it is now known that the formalism may be extended to apply to any systems integrable by the quantum  $R$ -matrix approach [8]. Much attention has been paid to the solutions of RE which present the boundary  $K$ -matrices compatible with the integrability. Recently, the boundary  $K$ -matrices have been constructed by several groups [6] for the Heisenberg spin- $\frac{1}{2}$  open chain and by the present author [7, 8] for the one-dimensional (1D) Hubbard open chain and for the 1D Bariev open chain.

On the other hand, the traditional basis for applying QISM to a completely integrable system is to represent the equations of motion of the system into Lax form. Following Korepin *et al* [9, 10], one may show that, for systems with periodic boundary conditions, the existence of the quantum  $R$ -matrix allows one to express the original equations of motion in Lax form. In particular, the Lax pairs for a variety of physically interesting models were given in [13–16]. Thus, one may expect that there is a variant of the usual Lax pair formulation to describe quantum integrable lattice open chains. Recently, we have shown that such a formulation does exist [17].

The aim of this letter is to present the Lax pair for the 1D small-polaron open chain in explicit form. From this the general boundary  $K$ -supermatrices are determined. Our construction provides a direct description for the quantum integrability of the system.

† E-mail address: cul@cbistic.sti.ac.cn

‡ Mailing address.

Let us first recall the Lax pair formulation for completely integrable lattice fermion open chains described in [17]. Instead of directly considering the equations of motion, let us study an operator version of an auxiliary linear problem:

$$\begin{aligned} \psi_{j+1} &= L_j(\lambda)\psi_j & j &= 1, 2, \dots, N \\ \frac{d}{dt}\psi_j &= M_j(\lambda)\psi_j & j &= 2, \dots, N \\ \frac{d}{dt}\psi_{N+1} &= \delta_N(\lambda)\psi_{N+1} & \frac{d}{dt}\psi_1 &= \delta'_1(\lambda)\psi_1. \end{aligned} \quad (1)$$

Here  $L_j$ ,  $M_j$ ,  $\delta$  and  $\delta'$  are some supermatrices depending on the spectral parameter  $\lambda$  and the dynamical variables. The consistency conditions for equation (1) yield the Lax equations

$$\begin{aligned} \frac{d}{dt}L_j(\lambda) &= M_{j+1}(\lambda)L_j(\lambda) - L_j(\lambda)M_j(\lambda) & j &= 2, \dots, N-1 \\ \frac{d}{dt}L_N(\lambda) &= \delta_N(\lambda)L_N(\lambda) - L_N(\lambda)M_N(\lambda) \\ \frac{d}{dt}L_1(\lambda) &= M_2(\lambda)L_1(\lambda) - L_1(\lambda)\delta'_1(\lambda). \end{aligned} \quad (2)$$

A lattice fermion open chain is said to be completely integrable if we can express the equations of motion in the Lax form (2), provided the boundary  $K$ -supermatrices exist as the solutions of equations (4) and (5) below. In fact, it is readily shown that a transfer matrix

$$\tau(\lambda) = \text{str}(K_+(\lambda)L_N(\lambda) \dots L_1(\lambda)K_-(\lambda)L_1^{-1}(-\lambda) \dots L_N^{-1}(-\lambda)) \quad (3)$$

does not depend on time  $t$ , provided the constrains hold:

$$K_-(\lambda)\delta'_1(-\lambda) = \delta'_1(\lambda)K_-(\lambda) \quad (4)$$

and

$$\text{str}[K_+(\lambda)\delta_N(\lambda)A_N(\lambda)] = \text{str}[K_+(\lambda)A_N(\lambda)\delta_N(-\lambda)] \quad (5)$$

with

$$A_N(\lambda) = L_N(\lambda) \dots L_1(\lambda)K_-(\lambda)L_1^{-1}(-\lambda) \dots L_N^{-1}(-\lambda). \quad (6)$$

Here the supertrace 'str' is taken over the auxiliary superspace. This implies that the system under study possesses an infinite number of conserved quantities.

Now, let us study the 1D small-polaron open chain with Hamiltonian

$$\begin{aligned} H = - \sum_{j=2}^N & [(a_j^\dagger a_{j-1} + a_{j-1}^\dagger a_j) + \frac{1}{2} \cos(2\eta)(2n_j - 1)(2n_{j-1} - 1)] + \sin(2\eta) \cot \xi_+ n_N \\ & + \alpha_+ a_N^\dagger + \beta_+ a_N + \sin(2\eta) \cot \xi_- n_1 + \alpha_- a_1^\dagger + \beta_- a_1. \end{aligned} \quad (7)$$

Here  $a_j^\dagger$  and  $a_j$  are, respectively, the creation and annihilation operators at lattice site  $j$ , and satisfy the usual anti-commutation relations

$$\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0 \quad \{a_j, a_k^\dagger\} = \delta_{jk}$$

and  $n_j$  is the density operator,  $n_j = a_j^\dagger a_j$ . Furthermore,  $\eta$  is a coupling parameter and  $\xi_\pm$ ,  $\alpha_\pm$  and  $\beta_\pm$  are some members of Grassmann algebra with  $\xi_\pm$  even and  $\alpha_\pm, \beta_\pm$  odd, satisfying  $\alpha_\pm \beta_\pm = 0$ .

It is not difficult to check that the equations of motion derived from the Hamiltonian (7) may be cast into the Lax form (2). Indeed, in our case, the  $L$  and  $M$  matrices take the form [16]

$$L_j(\lambda) = \begin{pmatrix} (i \sin(\lambda + 2\eta) - \sin \lambda)n_j + \sin \lambda & \sin 2\eta a_j \\ -i \sin 2\eta a_j^\dagger & (-i \sin \lambda - \sin(\lambda + 2\eta))n_j + \sin(\lambda + 2\eta) \end{pmatrix} \quad (8)$$

and

$$M_j(\lambda) = \begin{pmatrix} a_+ a_j^\dagger a_{j-1} - a_- a_j a_{j-1}^\dagger + a_0 a_j a_j^\dagger a_{j-1} a_{j-1}^\dagger & b_0(n_j a_{j-1} + i a_j n_{j-1}) + i b_- a_j + b_+ a_{j-1} \\ b_0(n_j a_{j-1}^\dagger - i a_j^\dagger n_{j-1}) - i b_+ a_j^\dagger + b_- a_{j-1}^\dagger & a_- a_j^\dagger a_{j-1} - a_+ a_j a_{j-1}^\dagger + a_0 n_j n_{j-1} \end{pmatrix} \quad (9)$$

with

$$\begin{aligned} a_+ &= - \left( i - \frac{\sin \lambda}{\sin(\lambda + 2\eta)} \right) & a_- &= - \left( i + \frac{\sin \lambda}{\sin(\lambda - 2\eta)} \right) \\ b_+ &= \frac{\sin 2\eta}{\sin(\lambda - 2\eta)} & b_- &= \frac{\sin 2\eta}{\sin(\lambda + 2\eta)} \\ a_0 &= \frac{i \sin 4\eta \sin 2\eta}{\sin(\lambda + 2\eta) \sin(\lambda - 2\eta)} & b_0 &= - \frac{\sin \lambda \sin 4\eta}{\sin(\lambda + 2\eta) \sin(\lambda - 2\eta)}. \end{aligned}$$

Here we emphasize that the  $L$  and  $M$  matrices are supermatrices with parities  $P(1) = 0$ ,  $P(2) = 1$ . From equation (2), it follows that

$$\delta'_1(\lambda) = \begin{pmatrix} \delta'_{11} & \delta'_{12} \\ \delta'_{21} & \delta'_{22} \end{pmatrix} \quad (10)$$

with

$$\delta'_{11} = \frac{1}{\Delta} \left[ \frac{i \sin^2 2\eta}{\sin \xi_-} \sin(2\eta + \xi_-)(1 - n_1) + \sin(\lambda - 2\eta)(i \sin(\lambda + 2\eta) - \sin \lambda) \alpha_- a_1^\dagger + \sin(\lambda + 2\eta)(i \sin(\lambda - 2\eta) + \sin \lambda) \beta_- a_1 \right]$$

$$\delta'_{12} = \frac{1}{\Delta} \left[ \frac{i \sin^2 2\eta}{\sin \xi_-} \sin(\lambda - \xi_-) a_1 - \sin 2\eta (\sin(\lambda + 2\eta) + \sin(\lambda - 2\eta)) n_1 \alpha_- + \sin 2\eta \sin(\lambda + 2\eta) \alpha_- \right]$$

$$\delta'_{21} = \frac{1}{\Delta} \left[ - \frac{i \sin^2 2\eta}{\sin \xi_-} \sin(\lambda + \xi_-) a_1^\dagger + \sin 2\eta (\sin(\lambda + 2\eta) + \sin(\lambda - 2\eta)) n_1 \beta_- - \sin 2\eta \sin(\lambda - 2\eta) \beta_- \right]$$

$$\delta'_{22} = \frac{1}{\Delta} \left[ - \frac{i \sin^2 2\eta}{\sin \xi_-} \sin(2\eta - \xi_-) n_1 + \sin(\lambda + 2\eta)(i \sin(\lambda - 2\eta) + \sin \lambda) \alpha_- a_1^\dagger + \sin(\lambda - 2\eta)(i \sin(\lambda + 2\eta) - \sin \lambda) \beta_- a_1 \right]$$

and

$$\delta_N(\lambda) = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} \quad (11)$$

with

$$\delta_{11} = \frac{1}{\Delta} \left[ \frac{i \sin^2 2\eta}{\sin \xi_+} \sin(2\eta + \xi_+)(1 - n_N) + \sin(\lambda + 2\eta)(i \sin(\lambda - 2\eta) + \sin \lambda) \alpha_+ a_N^\dagger \right. \\ \left. + \sin(\lambda - 2\eta)(i \sin(\lambda + 2\eta) - \sin \lambda) \beta_+ a_N \right]$$

$$\delta_{12} = \frac{1}{\Delta} \left[ \frac{\sin^2 2\eta}{\sin \xi_-} \sin(\lambda + \xi_+) a_N - i \sin 2\eta (\sin(\lambda + 2\eta) + \sin(\lambda - 2\eta)) n_N \alpha_+ \right. \\ \left. + i \sin 2\eta \sin(\lambda - 2\eta) \alpha_+ \right]$$

$$\delta_{21} = \frac{1}{\Delta} \left[ \frac{\sin^2 2\eta}{\sin \xi_+} \sin(\lambda - \xi_+) a_N^\dagger - i \sin 2\eta (\sin(\lambda + 2\eta) + \sin(\lambda - 2\eta)) n_N \beta_+ \right. \\ \left. + i \sin 2\eta \sin(\lambda + 2\eta) \beta_+ \right]$$

$$\delta_{22} = \frac{1}{\Delta} \left[ -\frac{i \sin^2 2\eta}{\sin \xi_+} \sin(2\eta - \xi_+) n_N + \sin(\lambda - 2\eta)(i \sin(\lambda + 2\eta) - \sin \lambda) \alpha_+ a_N^\dagger \right. \\ \left. + \sin(\lambda + 2\eta)(i \sin(\lambda - 2\eta) + \sin \lambda) \beta_+ a_N \right]$$

where  $\Delta = \sin(\lambda + 2\eta) \sin \lambda - 2\eta$ .

We now proceed to study the constraint conditions (4) and (5). Let us assume the boundary  $K$ -supermatrix  $K_-(\lambda)$  to take the form

$$K_-(\lambda) = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \quad (12)$$

and, substituting into equation (4), one may check that, out of the 16 homogeneous linear equations about  $K_{11}$ ,  $K_{12}$ ,  $K_{21}$  and  $K_{22}$ , only three are independent. After some algebraic calculations, we find

$$K_-(\lambda) = -\frac{1}{\sin \xi_-} \begin{pmatrix} \frac{\sin(\lambda - \xi_-)}{\sin \xi_-} & \frac{\alpha_- \sin \xi_- \sin 2\lambda}{i \sin 2\eta} \\ \frac{\beta_- \sin \xi_- \sin 2\lambda}{i \sin 2\eta} & -\sin(\lambda + \xi_-) \end{pmatrix} \quad (13)$$

(up to an unimportant scalar factor). In order to determine the boundary supermatrix  $K_+(\lambda)$ , let us first note that

$$A_N(\lambda) = L_N(\lambda) A_{N-1}(\lambda) L_N^{-1}(-\lambda). \quad (14)$$

Obviously, the matrix elements of  $A_{N-1}$  (anti)-commute with those of  $L_N$ . Keeping this fact in mind, and noting that the matrix elements of  $A_{N-1}$  are independent, we immediately obtain

$$K_+(\lambda) = \begin{pmatrix} \frac{\sin(\lambda + 2\eta - \xi_+)}{i \sin 2\eta} & -\frac{\alpha_+ \sin \xi_+ \sin 2(\lambda + 2\eta)}{i \sin 2\eta} \\ -\frac{\beta_+ \sin \xi_+ \sin 2(\lambda + 2\eta)}{i \sin 2\eta} & \sin(\lambda + 2\eta + \xi_+) \end{pmatrix} \quad (15)$$

(up to an unimportant scalar factor).

Now let us show that the Hamiltonian (7) may be related with the transfer matrix (3). Indeed, expanding the transfer matrix (3) in powers of  $\lambda$ , we have

$$\tau(\lambda) = \tau(0)[1 - \sin 2\eta(H + \text{constant})\lambda + \dots]. \quad (16)$$

Thus, we have shown that the model under consideration admits the Lax pair formulation.

In conclusion, we have presented the Lax pair for the 1D small-polaron open chain. The boundary  $K$ -supermatrices thus constructed should be the solutions of the graded version of the reflection equations. Thus our construction provides a basis for establishing the graded version of Sklyanin's formalism to describe integrable systems with boson and fermion fields in the finite interval. Here we emphasize that in contrast with the periodic case [16, 18], the Hamiltonian (7) may not be mapped into the 1D Heisenberg XXZ open chain via the Jordan–Wigner transformation. This implies that the system under consideration is essentially new. The extension of our construction to other open fermion chains, such as the 1D Hubbard open chain [7] and the 1D Bariev open chain [8], is also interesting. We hope to return to these questions in the near future.

This work was supported in part by the National Natural Science Foundation of China under grant No 19505009. I am grateful to Professor Xing-Chang Song and Professor Chong-Sheng Li for their support and encouragement.

## References

- [1] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **A 21** 2375
- [2] Mezincescu L, Nepomechie R I and Rittenberg V 1990 *Phys. Lett.* **147A** 70
- [3] Pasquier V and Saleur H 1990 *Nucl. Phys. B* **330** 523
- [4] Destri C and de Vega H J 1992 *Nucl. Phys. B* **37** 692
- [5] Mezincescu L and Nepomechie R I 1991 *J. Phys. A: Math. Gen.* **24** L17
- [6] de Vega H J and González Ruiz A 1991 *J. Phys. A: Math. Gen.* **26** L519  
Hou B Y and Yue R H 1993 *Phys. Lett.* **183A** 169  
Inami T and Konno H 1994 *J. Phys. A: Math. Gen.* **27** L913
- [7] Zhou H Q 1996 *Phys. Rev. B* **54** 41
- [8] Zhou H Q 1996 *Phys. Rev. B* at press
- [9] Korepin V E, Izergin G and Bogoliubov N M 1992 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
- [10] Faddeev L D 1984 *Les Houches 1982* ed J B Zuber and R Stora (Amsterdam: North-Holland); 1980 *Sov. Sci. Rev. Math. Phys. C* **1** 107  
Kulish P P and Sklyanin E K 1982 *Lecture Notes in Phys.* **151** 61  
Izergin A G and Korepin V E 1982 *Sov. J. Part. Nucl. Phys.* **13** 20
- [11] Thacker H B 1982 *Rev. Mod. Phys.* **53** 253
- [12] Cherednik I 1984 *Theor. Math. Phys.* **61** 35
- [13] Sogo K and Wadati M 1982 *Prog. Theor. Phys.* **68** 85
- [14] Barouch E 1984 *Stud. Appl. Math.* **70** 151
- [15] Wadati M, Olmedilla E and Akutsu Y 1987 *J. Phys. Soc. Japan* **56** 1340
- [16] Zhou H Q, Jiang L J and Tang J G 1990 *J. Phys. A: Math. Gen.* **23** 213
- [17] Zhou H Q 1995 *Preprint* CQU-TH-95-18
- [18] Zhou H Q, Jiang L J and Wu P F 1989 *Phys. Lett.* **137A** 224  
Pu F C and Zhao B H 1986 *Phys. Lett.* **118A** 77